Solutions to Problem Set #2: Phase planes and bifurcation diagrams

Due: Friday October 13, 2006

Consider the following models:

Model 1:

$$\frac{dx}{dt} = -k_1 x + k_2 \frac{y}{k_y + y^2} \tag{1a}$$

$$\frac{dy}{dt} = -k_1 y + k_2 \frac{x}{k_x + x^2} \tag{1b}$$

Model 2:

$$\frac{dx}{dt} = k_1 x - k_2 y + k_3 x y^2 \tag{2a}$$

$$\frac{dy}{dt} = k_4 x + k_1 y + k_3 y^3 \tag{2b}$$

Model 3:

$$\frac{dx}{dt} = k_4 A - k_1 x - k_2 x + k_3 x^2 y \tag{3a}$$

$$\frac{dy}{dt} = k_1 x - k_3 x^2 y \tag{3b}$$

Choose two of the three models and investigate them using the tools developed in class.

Please hand in (where applicable):

- The non-dimensional model, indicating all characteristic scales and parameter groupings
- Phase portraits for several choices of parameters, indicating interesting features and an interpretation of the solutions in terms of the dynamics of the original system
- Bifurcation diagrams with respect to parameters of interest (bifurcation parameters)
- Eigenvalue plots with respect to the bifurcation parameter, to explain what's happening at the bifurcation and why

- Two parameter continuation plots

Solutions

Model 1:
For the non-dimensionalization, choose the following characteristic scales:

$$\tau = \frac{1}{k_1},\ \bar{x} = \sqrt{k_x},\ \bar{y} = \sqrt{k_y},$$

giving the non-dimensionalized equations:

$$\frac{dx}{dt} = -x + \lambda \frac{y}{1+y^2} \quad (4)$$

$$\frac{dy}{dt} = -y + \lambda \frac{x}{1+x^2} \quad (5)$$

where $\lambda = \frac{k_2}{k_1\sqrt{k_x k_y}}$.

This system has one or three steady states depending on the value of λ:

- If $|\lambda| < 1$, there is one steady state at (0,0)
- If $\lambda \geq 1$, there are three steady states at (0,0), $(\sqrt{\lambda - 1}, \sqrt{\lambda - 1})$, $(-\sqrt{\lambda - 1}, -\sqrt{\lambda - 1})$
- If $\lambda \leq 1$, there are three steady states at (0,0), $(\sqrt{-\lambda - 1}, -\sqrt{-\lambda - 1})$, $(-\sqrt{-\lambda - 1}, \sqrt{-\lambda - 1})$

Using the Jacobian to determine stability, at (0,0) we have

$$J = \begin{pmatrix} -1 & \lambda \\ \lambda & -1 \end{pmatrix}$$

so that

$$Tr(J) = -2 < 0 \quad (6)$$

$$Det(J) = 1 - \lambda^2 \quad (7)$$

$$Discr(J) = 4\lambda^2 > 0. \quad (8)$$

Based on the values of the Determinant, the steady state at (0,0) is a stable node if $|\lambda| < 1$ and a saddle node otherwise.

Similarly, $(\sqrt{\lambda - 1}, \sqrt{\lambda - 1})$, $(-\sqrt{\lambda - 1}, -\sqrt{\lambda - 1})$ are stable nodes if $\lambda > 1$ and $(\sqrt{-\lambda - 1}, -\sqrt{-\lambda - 1})$, $(-\sqrt{-\lambda - 1}, \sqrt{-\lambda - 1})$ are stable nodes if $\lambda < -1$.

The phase portraits for $\lambda = -2, 0, +2$ are shown in Figure 1, the bifurcation diagram is shown in Figure 2 and an eigenvalue plot is shown in Figure 3. In Figure 3, only the eigenvalues of the steady state at (0,0) is shown since the other steady states are always stable nodes, given the appropriate value of λ. At (0,0), we see the Trace of the Jacobian is always negative and the Determinant changes sign, causing the steady state to transition from a saddle node for low λ values, to a stable node for $|\lambda| < 1$ and back to a saddle node for high values of λ. By considering the bifurcation diagram (Figure 2), we see the emergence of two new steady states as (0,0) changes stability from saddle to stable node. This type of bifurcation is called a Supercritical Pitchfork at $\lambda = 2$ and a Subcritical Pitchfork at $\lambda = -2$.

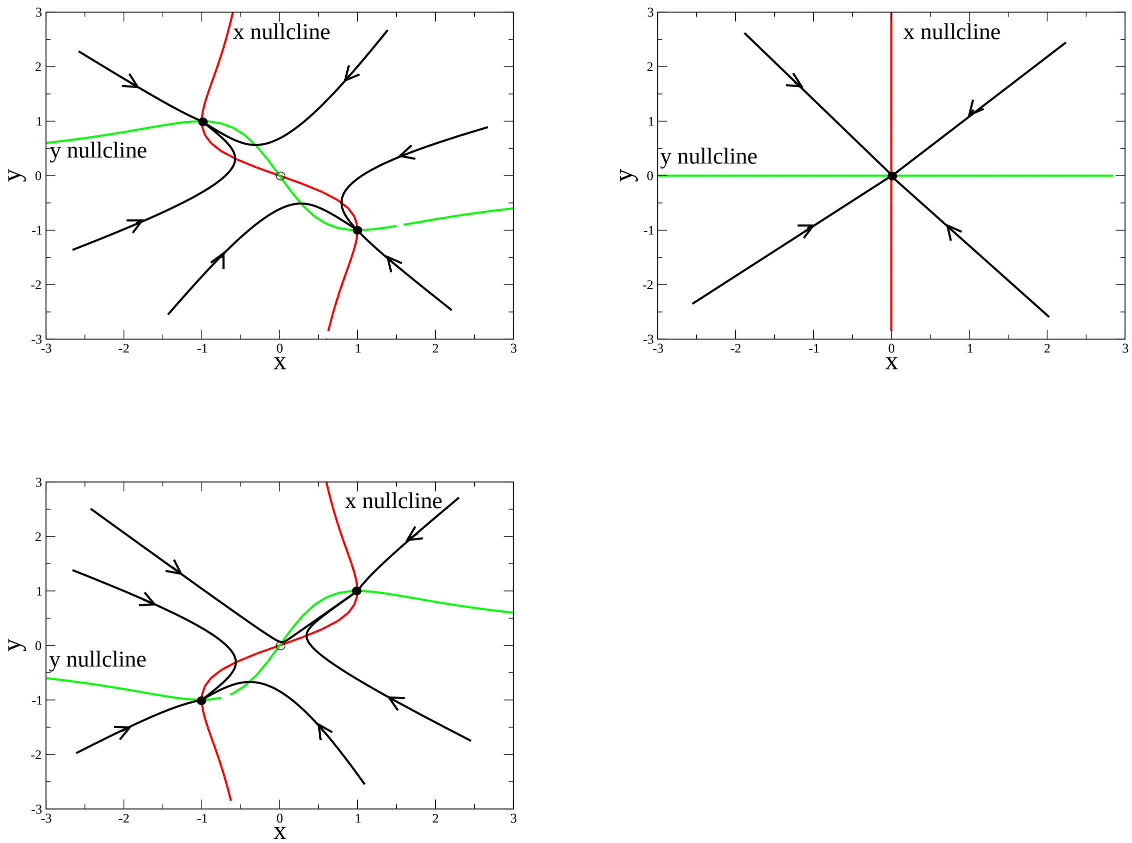

Figure 1: Phase portraits and nullclines for Model 1 with $\lambda = -2, 0, 2$ (top left, top right and bottom, respectively).

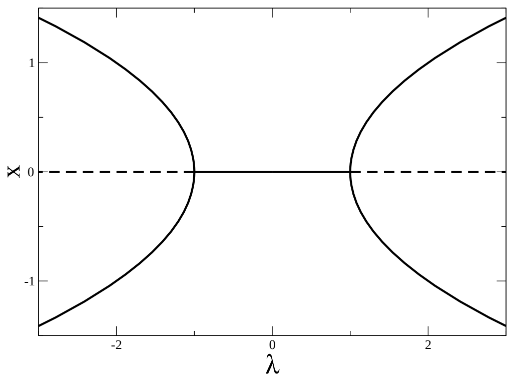

Figure 2: Bifurcation diagram for Model 1.

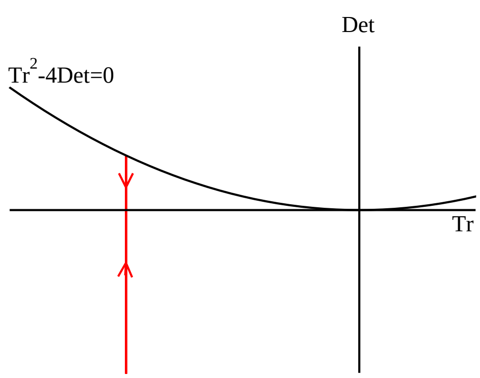

Figure 3: Eigenvalue plot for Model 1.

Model 2:
For the non-dimensionalization, choose the following characteristic scales:

$$\tau = \frac{1}{\sqrt{k_2 k_4}},\ \bar{x} = \sqrt{\frac{k_2}{k_4}}\sqrt{\frac{\sqrt{k_2 k_4}}{k_3}},\ \bar{y} = \sqrt{\frac{\sqrt{k_2 k_4}}{k_3}},$$

giving the non-dimensionalized equations:

$$\frac{dx}{dt} = \mu x - y + xy^2 \quad (9)$$

$$\frac{dy}{dt} = x + \mu y + y^3 \quad (10)$$

where $\mu = \frac{k_1}{\sqrt{k_2 k_4}}$.

This only has one real-valued steady state at (0,0). Using the Jacobian to determine stability, at (0,0) we have

$$J = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

so that

$$Tr(J) = 2\mu \quad (11)$$

$$Det(J) = \mu^2 + 1 > 0 \quad (12)$$

$$Discr(J) = -4 < 0. \quad (13)$$

Based on the values of the Determinant, the steady state at (0,0) is a stable spiral if $\mu < 0$ and an unstable spiral otherwise. We can verify using the Hopf Bifurcation Theorem that we have a Hopf bifurcation at the critical value of μ: $\mu_c = 0$.

The phase portraits for $\mu = -1, +1$ are shown in Figure 4, the bifurcation diagram is shown in Figure 5 and an eigenvalue plot is shown in Figure 6. We can see from Figure 6 that the eigenvalues are always complex and cross the Determinant axis when $\mu = 0$, consistent with a Hopf bifurcation. From the bifurcation diagram (Figure 5), we see there is an unstable limit cycle for $\mu < 0$ and this is verified by the phase portraits (Figure 4). This type of bifurcation is called a Subcritical Hopf.

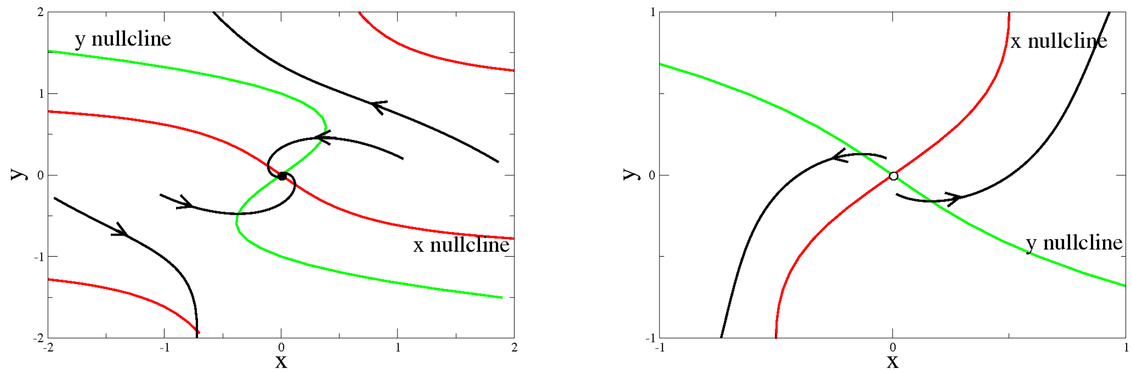

Figure 4: Phase portraits and nullclines for Model 2 with $\mu = -1, 1$ (left and right, respectively).

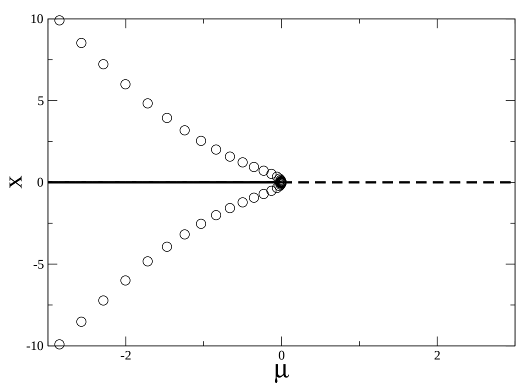

Figure 5: Bifurcation diagram for Model 2.

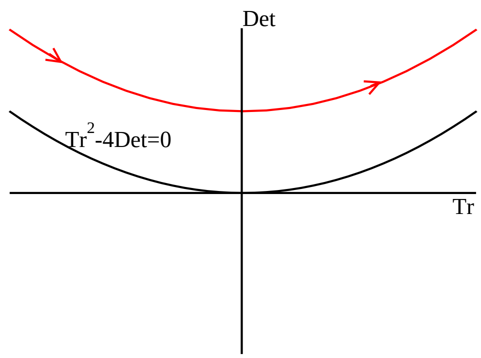

Figure 6: Eigenvalue plot for Model 2.

Model 3: For the non-dimensionalization, choose the following characteristic scales:

$$\tau = \frac{1}{k_2},\ \bar{x} = \sqrt{\frac{k_2}{k_3}},\ \bar{y} = \sqrt{\frac{k_2}{k_3}},$$

giving the non-dimensionalized equations:

$$\frac{dx}{dt} = a - (b+1)x + x^2 y \tag{14}$$

$$\frac{dy}{dt} = bx - x^2 y \tag{15}$$

where $a = \frac{k_4\sqrt{k_3}A}{k_2\sqrt{k_2}}$, $b = \frac{k_1}{k_2}$.

This only has one real-valued steady state at $(a, b/a)$. Using the Jacobian to determine stability, at $(a, b/a)$ we have

$$J = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}$$

so that

$$Tr(J) = b - 1 - a^2 \tag{16}$$

$$Det(J) = a^2 > 0 \tag{17}$$

$$Discr(J) = (b - 1 - a^2)^2 - 4a^2. \tag{18}$$

Based on the values of the Trace and Discriminant, there are four possibilities for the stability of $(a, b/a)$:

- Stable node if $b < 1 + a^2$ and $b > (1+a)^2$
- Stable spiral if $b < 1 + a^2$ and $b < (1+a)^2$
- Unstable spiral if $b > 1 + a^2$ and $b < (1+a)^2$
- Unstable node if $b > 1 + a^2$ and $b > (1+a)^2$

Looking at the conditions for the steady state to be a stable or unstable spiral, it suggests there is a critical point $b_c = 1 + a^2$ where a Hopf bifurcation can occur. Using the Hopf Bifurcation Theorem, we can prove this is the case.

The phase portraits for $a = 1$, $b = 1, 3$ are shown in Figure 7, the bifurcation diagram is shown in Figure 8, a two parameter continuation plot is shown in Figure 9 and an eigenvalue plot is shown in Figure 10. We can see from Figure 10 that the eigenvalues can be complex and cross the Determinant axis when $b = 1 + a^2$, consistent with a Hopf bifurcation. As we increase b further, the unstable spiral turns into an unstable node.

From the bifurcation diagram (Figure 8), we see there is an stable limit cycle for $b > 1 + a^2$ and this is verified by the phase portraits (Figure 7). This type of bifurcation is called a Supercritical Hopf and this model is the space-free Brusselator which we'll look at again in Problem Set #3.

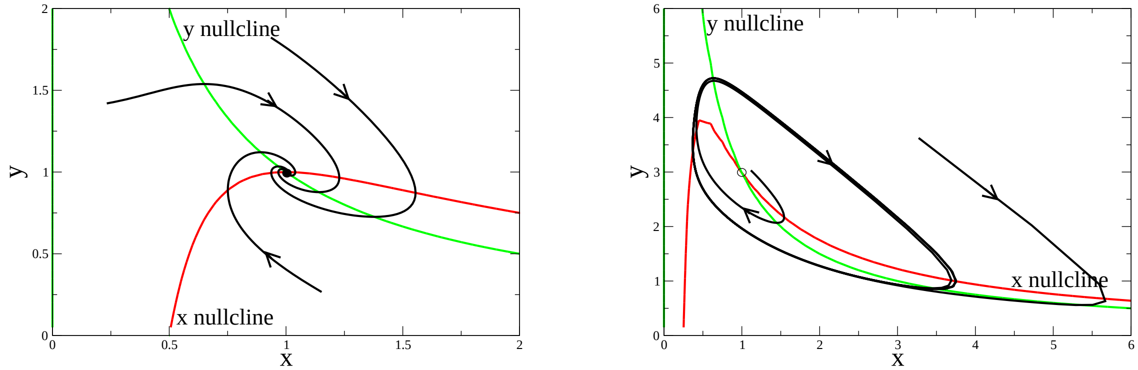

Figure 7: Phase portraits and nullclines for Model 3 with $a = 1$ and $b = 1, 3$ (left and right, respectively).

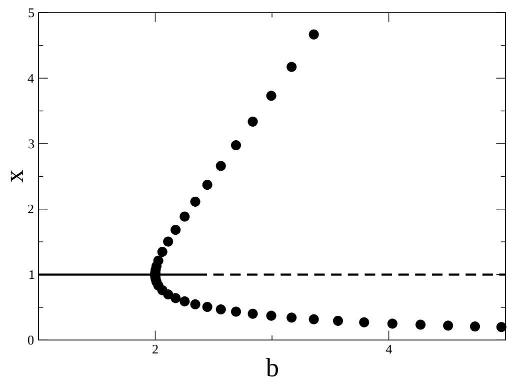

Figure 8: Bifurcation diagram for Model 3.

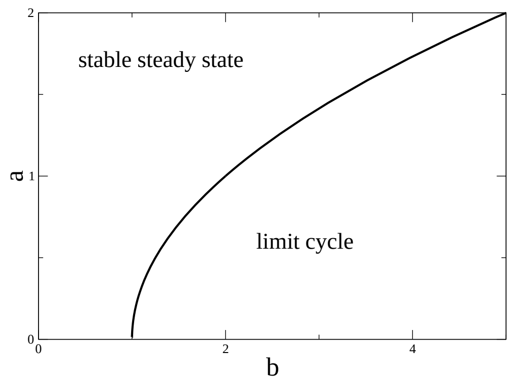

Figure 9: Two parameter continuation plot for Model 3.

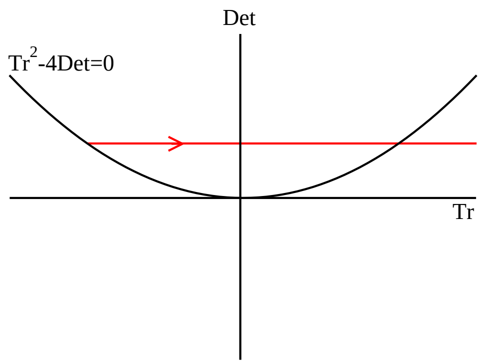

Figure 10: Eigenvalue plot for Model 3.